A Hopefully Self-contained Introduction to Affine Planes

Andres Duarte

Contents

Introduction

In this paper we define abstract affine planes, prove some theorems about them. and give three different models of affine planes: The classical affine plane, an affine plane from a division ring, and a Moulton plane.

Abstract affine plane

Suppose $\mathcal{P}$ is a set of *points* and $\mathcal{L}$ is a set of *lines*. If l and m are lines, then l is *parallel* to m, denoted $l \parallel m$, if and only if they are equal or disjoint.

$$l \parallel m \leftrightarrow l = m \text{ or } l \cap m = \varnothing$$

$(\mathcal{P}, \mathcal{L})$ is an *affine plane* if and only if it satifies the following axioms

(A1) For any two distinct points p, q there is a unique line $\ell(p, q)$ that contains them.

$$\forall p, q \in \mathcal{P}\ \exists! \ell(p, q) \in \mathcal{L} \quad p, q \in \ell(p, q)$$

(A2) For any line l and any point p, there is a unique line $\ell(p \parallel l)$ that contains p and is parallel to l.

$$\forall l \in \mathcal{L}\ \forall p \in \mathcal{P}\ \exists! \ell(p \parallel l) \in \mathcal{L} \quad p \in \ell(p \parallel l) \text{ and } \ell(p \parallel l) \parallel l$$

(A3) There exists three points such that no line contains three of them.

$$\exists a, b, c \in \mathcal{P} \quad \nexists l \in \mathcal{L}\ a, b, c \in l$$

Theorem 1. *Any two lines in an affine plane are either parallel or their intersection is a singleton.*

Proof. Suppose l_1 and l_2 are distinct lines in an affine plane and $|l_1 \cap l_2| > 1$. Then there exists $p, q \in l_1 \cap l_2$ distinct, a contradiction to the uniqueness of $\ell(p, q)$. □

Theorem 2. *Parallelism is an equivalence relation on the lines of the affine plane.*

Proof. Parallelism is clearly reflexive and symmetric. To prove that it is transitive suppose l_1, l_2, l_3, l_4 are lines in an affine plane, $l_1 \parallel l_2$, and $l_2 \parallel l_3$

(l_1, l_2, l_3 not distinct) Immediatly $l_1 \parallel l_3$.

(l_1, l_2, l_3 distinct) Suppose for a contradiction that $l_1 \nparallel l_3$. Then $\exists p \in \mathcal{P} \quad l_1 \cap l_3 = \{p\}$, a contradiction to the uniqueness of $\ell(p \parallel l_2)$. Therefore $l_1 \parallel l_3$.

□

Lemma 3. *For $\sim$ an equivalence relation*

$$A \sim B \nsim C \to A \nsim C$$

Proof. Assume $A \sim B \nsim C$ and suppose for a contradiction $A \sim C$. By symmetry $B \sim A$. By transitivity $B \sim C$, a contradiction. Hence $A \nsim C$. □

Theorem 4. *There is a bijection between any two lines in an affine plane.*

Proof. Suppose l and l' are distinct lines in an affine plane. Then $\exists p \in l \ \ p \notin l'$ and $\exists p' \in l' \ \ p' \notin l$. Suppose $r \in l$. Then by construction

$$\ell(r \parallel \ell(p, p')) \parallel \ell(p, p') \nparallel l'$$

Thus by Lemma 3 $\ell(r \parallel \ell(p, p')) \nparallel l'$. Hence by theorem 1

$$\exists h \in \mathcal{P} \quad \ell(r \parallel \ell(p, p')) \cap l' = \{h\}$$

For all $r \in l$ define f by $r \mapsto h$. Repeating the above argument interchanging l and l' gives you f^{-1}. □

Affine plane from a field

Let $\mathbb{F}$ be a field and define the following

$$\begin{aligned}\langle m,b\rangle &:= \{(x,mx+b) : x\in\mathbb{F}\}\\ \langle a\rangle &:= \{(a,y) : y\in\mathbb{F}\}\\ \mathcal{L}_{\mathbb{F}} &:= \{\langle m,b\rangle : m,b\in\mathbb{F}\}\cup\{\langle a\rangle : a\in\mathbb{F}\}\end{aligned}$$

Then the *classical affine plane of dimension 2*, denoted $\mathbb{A}^2_{\mathbb{F}}$, is

$$\mathbb{A}^2_{\mathbb{F}} := (\mathbb{F}^2, \mathcal{L}_{\mathbb{F}})$$

Lemma 5. $\langle m,b\rangle || \langle n,c\rangle \leftrightarrow m=n$

Proof.

($\rightarrow$) We argue by contraposition. Suppose $m\neq n$, then

$$\begin{aligned}mx+b=nx+c &\leftrightarrow mx-nx=c-b\\ &\leftrightarrow x(m-n)=c-b\\ &\leftrightarrow x=\frac{c-b}{m-n}\\ &\rightarrow |\langle m,b\rangle\cap\langle n,c\rangle|=1\\ &\rightarrow \langle m,b\rangle \nparallel \langle n,c\rangle\end{aligned}$$

Hence, by contraposition $\langle m,b\rangle||\langle n,c\rangle \rightarrow m=n$.

($\leftarrow$)

($b=c$) Immediatly $\langle m,b\rangle=\langle n,c\rangle$

($b\neq c$) Suppose for a contradiction there exists $(x,y)\in\langle m,b\rangle\cap\langle n,c\rangle$, then

$$\begin{aligned}y=mx+b \text{ and } y=nx+c &\rightarrow mx-nx+b-c=0\\ &\rightarrow x(m-n)+b-c=0\\ &\rightarrow b=c,\ \text{ a contradiction} \qquad ↯\end{aligned}$$

Hence, $\langle m,b\rangle||\langle n,c\rangle \leftrightarrow m=n$ as desired ☐

Lemma 6. $\forall a,b\in\mathbb{F}\ \ \langle a\rangle \parallel \langle b\rangle$

Proof.

$$\begin{aligned}a=b &\rightarrow \langle a\rangle=\langle b\rangle\\ &\rightarrow \langle a\rangle \parallel \langle b\rangle\\ a\neq b &\rightarrow \langle a\rangle\cap\langle b\rangle=\varnothing\\ &\rightarrow \langle a\rangle \parallel \langle b\rangle\end{aligned}$$

☐

Lemma 7 (A1). *For any two distinct points $p, q \in \mathbb{F}^2$ there is a unique line $l \in \mathcal{L}_{\mathbb{F}}$ that contains them.*

Proof. Consider $(x_0, y_0), (x_1, y_1) \in \mathbb{F}^2$ and the line

$$l := \left\{ \left(x, \frac{y_0 - y_1}{x_0 - x_1} x + \frac{x_0 y_1 - x_1 y_0}{x_0 - x_1} \right) : x \in \mathbb{F} \right\}$$

If $x_0 = x_1$, then $(x_0, y_0), (x_1, y_1) \in \{(x_0, y) : y \in \mathbb{F}\}$. If $x_0 \neq x_1$, then

$$\begin{aligned}
mx_0 + b &= \frac{y_0 - y_1}{x_0 - x_1} x_0 + \frac{x_0 y_1 - x_1 y_0}{x_0 - x_1} \\
&= \frac{x_0 y_0 - x_0 y_1 + x_0 y_1 - x_1 y_0}{x_0 - x_1} \\
&= \frac{x_0 - x_1}{x_0 - x_1} y_0 \\
&= y_0 \\
&\to (x_0, y_0) \in l \\
mx_1 + b &= \frac{y_0 - y_1}{x_0 - x_1} x_1 + \frac{x_0 y_1 - x_1 y_0}{x_0 - x_1} \\
&= \frac{x_1 y_0 - x_1 y_1 + x_0 y_1 - x_1 y_0}{x_0 - x_1} \\
&= \frac{x_0 - x_1}{x_0 - x_1} y_1 \\
&= y_1 \\
&\to (x_1, y_1) \in l
\end{aligned}$$

Let $l' := \{(x, nx + c) : x \in \mathbb{F}\}$ and suppose $(x_0, y_0), (x_1, y_1) \in l'$, then

$$\begin{aligned}
mx_0 + b = nx_0 + c \text{ and } mx_1 + b = nx_1 + c &\to mx_0 - mx_1 = nx_0 - nx_1 \\
&\to m(x_0 - x_1) = n(x_0 - x_1) \\
&\to m = n \\
&\to mx_0 + b = mx_0 + c \\
&\to b = c \\
&\to l = l'
\end{aligned}$$

□

Lemma 8 (A2). *For any line $l \in \mathcal{L}_{\mathbb{F}}$ and any point $(x_0, y_0) \in \mathbb{F}^2$, there is a unique line $l' \in \mathcal{L}_{\mathbb{F}}$ that contains p and is parallel to l.*

Proof. Suppose $l = \langle m, b\rangle$, then $\langle m, y_0 - mx_0\rangle$ contains (x_0, y_0) and is parallel to $\langle m, b\rangle$ by Lemma 5. Suppose $\langle m, b'\rangle$ also contains (x_0, y_0), then

$$\begin{aligned} y_0 = mx_0 + y_0 - mx_0 \text{ and } y_0 = mx_0 + b' &\to y_0 - mx_0 = b' \\ &\to \langle m, b'\rangle = \langle m, y_0 - mx_0\rangle \end{aligned}$$

Suppose instead $l = \langle a\rangle$. Then $\langle x_0\rangle$ is parallel to $\langle a\rangle$ by Lemma 5 and contains (x_0, y_0) and is by construction the only such line. ☐

Lemma 9 (A3)**.** *There exists three points $p, q, r \in \mathbb{F}^2$ such that no line $l \in \mathcal{L}_{\mathbb{F}}$ contains them.*

Proof. Take $p = (0, 0)$, $q = (0, 1)$ and $r = (1, 0)$, then $\langle 0\rangle$ is the unique line containing p and q. and $r \notin \langle 0\rangle$. Hence no line can contain all three points. ☐

Theorem 10. *$\mathbb{A}^2_{\mathbb{F}}$ is an affine plane.*

Proof. Immediate from lemmas, 8, 9, and 10. ☐

Affine plane from a division ring

If S is a set and $+$ and $\cdot$ are binary operations on S. Then $(S, +, \cdot)$ is a *ring* if it satisfies the folowing axioms

(R1) $(S, +)$ is an abelian group.

(R2) $(S, \cdot)$ is a monoid (multiplication is associate and there exists a multiplicative identity).

(R3) multiplication distributes over addition.

$(S, +, \cdot)$ is a *division ring* if and only if $(S, +, \cdot)$ is a ring and every nonzero element of S has a multiplicative inverse. Let $(K, +, \cdot)$ be a division ring and define

$$L_K := \{a + Kb : a, b \in K^2, b \neq 0\}$$

Lemma 11. *Let $l_1 = a + Kb$ and $l_2 = c + Kd$, then*

1. *If b and d are linearly independent, then $l_1 \cap l_2$ is a singleton.*
2. *If b and d are lineraly dependent, then $l_1 \parallel l_2$. In particular, if $p - a \notin Kb$, then $l_1 \cap l_2 = \varnothing$. Otherwise $l_1 = l_2$.*

Proof.

1. Suppose b and d are linearly independent. Then $\{b, d\}$ is a basis of K^2 and there exists unique $x, y \in K$ such that $c - a = xb + yd$

$$\begin{aligned} c - a = xb + yd &\leftrightarrow a + xb = c - yd \\ &\to l_1 \cap l_2 = \{a + xb\} \end{aligned}$$

2. Suppose b and d are linearly dependent. Then there exists $x, y \in K$ not all zero such that $xb + yd = 0$. Without loss of generality assume that y is nonzero, then x must also be nonzero, because by construction b and d are nonzero and division rings don't have zero divisors. Then

$$\begin{aligned} l_2 &= c + kd \\ &= c + K(-y)^{-1}xb \\ &= c + Kb \end{aligned}$$

$$\begin{aligned} c - a \notin Kb \to \forall t, u \in K \quad (t-u)b &\neq c - a \\ a + tb &\neq c + ub \\ \to l_1 \cap l_2 &= \varnothing \end{aligned}$$

$$\begin{aligned} c - a \in Kb \to \exists \delta \in K \quad c - a &= \delta b \\ c &= a + \delta b \\ \to l_2 &= a + \delta b + Kb \\ &= a + Kb \\ &= l_1 \end{aligned}$$

□

Corollary 11.1. *For any $l_1, l_2 \in \mathcal{L}_K$ either $l_1 \parallel l_2$ or $l_1 \cap l_2$ is a singleton.*

Lemma 12 (A1)**.** *For any two distinct points $p, q \in K^2$ there is a unique line $l \in L_K$ that contains them.*

Proof. Suppose $p, q \in K^2$ are distinct and $l = p + K(q - p)$. Then

$$p = p + 0(q - p) \text{ and } q = q + 0(q - p) \to p, q \in l$$

Suppose $\exists l' \in L_K \quad p, q \in l'$, then by Corollary 11.1 $l \parallel l'$ and in particular $l = l'$. □

Lemma 13 (A2)**.** *For any line $l \in \mathcal{L}_K$ and any point $p \in K^2$ there is a unique line that contains p and is parallel to l.*

Proof. Suppose $p \in K^2$, $l = x + Ky$, and $l' = p + Ky$. Then $l \parallel l'$ by Lemma 11. And $p = p + 0y$ implies $p \in l'$. □

Lemma 14 (A3)**.** *There exists three points $p, q, r \in K^2$ such that no line contains all three of them .*

Proof. Suppose $p = (0, 0), q = (0, 1)$, and $r = (1, 0)$. Then $l = 0 + Kq$ is the unique line containing p and q. □

Theorem 15. *$(K^2, \mathcal{L}_K)$ is an affine plane*

Proof. Immediate from lemmas 12, 13, and 14. □

Moulton plane

Define $\odot : \mathbb{R}^2 \to \mathbb{R}$ by

$$m \odot x \begin{cases} mx & \text{if } m \leq 0 \text{ or } x \leq 0 \\ 2mx & \text{otherwise} \end{cases}$$

and let

$$\mathcal{L}_M := \{\{(a, y) : y \in \mathbb{R}\} : a \in \mathbb{R}\} \cup \{\{(x, m \odot x + b) : x \in R\} : m, b \in \mathbb{R}\}$$

Then the *Moulton plane* is $(\mathbb{R}^2, \mathcal{L}_M)$.

Lemma 16 (A1)**.** *For any two distinct points $p, q \in \mathbb{R}^2$ there is a unique line $l \in \mathcal{L}_M$ that contains them.*

Proof. Let $p = (x_0, y_0)$ and $q = (x_1, y_1)$ The only new cases are those where p and q lie on different sides of the y axis and $y_0 < y_1$.

$$\begin{aligned} y_0 = mx_0 + b \text{ and } y_1 = 2mx_1 + b \to y_0 - y_1 &= mx_0 - 2mx_1 \\ &= m(x_0 - 2x_1) \\ m &= \frac{y_0 - y_1}{x_0 - 2x_1} \\ \to b &= y_0 - \frac{y_0 - y_1}{x_0 - 2x_1} x_0 \end{aligned}$$

$$\to p, q \in \{(x, \frac{y_0 - y_1}{x_0 - 2x_1} \circ x + y_0 - \frac{y_0 - y_1}{x_0 - 2x_1} x_0) : x \in \mathbb{R}\}$$

□

Lemma 17 (A2)**.** *For any line $l \in \mathcal{L}_M$ and any point $p \in \mathbb{R}^2$ there is a unique line that contains p and is parallel to l.*

Proof. Suppose $l = \{(x, m \odot x + b) : x \in \mathbb{R}\}$, $p = (x_0, y_0)$, and

$$l' = \{(x, m \odot x + y_0 - m \odot x_0) : x \in \mathbb{R}\}$$

Then $l' \parallel l$ and $p \in l'$. Suppose $l'' = \{(x, m \odot x + b') : x \in \mathbb{R}\}$, $l'' \parallel l$ and $p \in l''$. Then

$$y_0 = m \odot x_0 + y_0 - m \odot x_0 \text{ and } y_0 = m \odot x_0 + b'$$

$$\begin{aligned} \to m \odot x_0 + y_0 - m \odot x_0 &= m \odot x_0 + b' \\ \to y_0 - m \odot x_0 &= b' \\ \to l' &= l'' \end{aligned}$$

□

Lemma 18 (A3)**.** *There exists three points* $p, q, r \in \mathbb{R}^2$ *such that no line* $l \in \mathcal{L}_M$ *contains three of them.*

Proof. The proof is equivalent to the proof of Lemma 9. □

Theorem 19. $(\mathbb{R}^2, \mathcal{L}_M)$ *is an affine plane.*

Proof. Immediate from lemmas 16, 17, and 18. □

References

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